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INFLUENCE OF HIGH HYDROSTATIC PRESSURE EXTRUSION ON MECHANICAL BEHAVIOR OF MATERIALS

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13. ABSTRACT

Hydrostatic extrusion is a promising new metal working process. It may provide a practical technique to form useful wrought products from materials difficult or impossible to work by other processes, and it offers the means to develope superior useful properties. This program of research is concerned with the response of a variety of materials to hydrostatic extrusion. A major objective is to relate the microstructure and mechanical properties of extruded materials to the important processing variables, extrusion ratio, extrusion pressure, temperature, and extrusion rate. The experiments fall into two categories; 1. A detailed, systematic investigation of the influence of the processing variables on the properties of several fairly simple materials (pure Fe, Ni, Mg, and Ti: and Cu-30Zn or Cu-10Sn) 2. The utilization of hydrostatic extrusion to optimize the properties of more complex materials chosen for their technological utility (Mg-Li-B for high specific stiffness; Fe-C for high strength, ductility; T-D Nichrome for high temperature strength; Al-Fe, a high conductivity, low density material; NbTi a super conducting alloy). The theoretical portion of the program is devoted to an understanding of extrusion. Elastic-plastic finite-element programs are being considered as a means of analyzing the extrusion process. Such studies are directed towards a complete stress analysis of extrusion and should lead to a better elucidation of the factors influencing the onset of undesirable defects during extrusion.

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INFLUENCE OF HIGH HYDROSTATIC PRESSURE EXTRUSION
ON
MECHANICAL BEHAVIOR OF MATERIALS

Introduction and Summary

Hydrostatic extrusion is a promising new metal working technique. In hydrostatic extrusion a large hydrostatic pressure is superimposed on the stress field that normally develops during extrusion. As a result, materials difficult or impossible to extrude by the conventional extrusion process can be extruded hydrostatically. Further, there is evidence that materials formed by hydrostatic extrusion exhibit superior mechanical properties compared to similar materials worked by conventional techniques. At the present time hydrostatic extrusion is not being applied extensively to technological applications in this country (considerably more is being done in Europe and Japan). A pilot plant to extrude hydrostatically copper into wire has been started by Western Electric and this particular operation appears very promising. Surprisingly the basic understanding of the hydrostatic extrusion process, both from the theoretical mechanics point of view and from the standpoint of materials science, remains in quite a primitive state.

The present program is centered on an experimental and analytical study of hydrostatic extrusion. The hydrostatic extrusion press and auxiliary equipment arrived in July 1975 and was made operational by late August 1975. The experimental aspects are described in the first section of this progress report (pages 2-11). In the second section (pages 12-31) we describe the basis for an elastic-plastic code as a means of assessing metal forming operations. This analytical approach is unique and promises to be a very powerful tool in analyzing the extrusion process.

I. Experimental Aspects of Hydrostatic Extrusion Program

(Robert Whalen, Research Associate and Larry Eiselstein, Research Assistant)

After a series of delays caused by a variety of factors such as machining and supply problems, modifications to the design, and an overly optimistic original promise date, the hydrostatic extrusion machine was finally delivered to Stanford University in June of 1975. The major thrust of our activities at Stanford since that date has been on the installation and investigatory testing of the extruder. The machine's special power requirements necessitated the re-wiring of a portion of the John Blume Structures Laboratory in which the extruder is located. An investigation of alternatives in this area yielded two options. The first of these was to run an 80 Amp capacity, three phase line to the hydrostatic extruder and operate as a 220 volt system. The second option was to utilize a step-up transformer which would provide 440 volts at the main box and require the installation of a 40 Amp, three line. Fortunately, we discovered a surplus power transformer on campus and this brought estimates for each of the two possibilities to approximately the same cost. There was another consideration which influenced our thinking and finally proved persuasive. The extruder already shares the Structures Lab with other heavy power consuming machinery and computers. As the laboratory expands there should be a definite advantage to maintaining a distinct power system which would be independent of power demands and fluctuations caused by these competing users. Therefore the decision was made to install the surplus transformer at the main line and run the extrusion project on 440 volts.

Location of the machine in the Structures Lab is adequate for our purposes. The site provides sufficient work space for our purposes and the Structures Lab is close to other essential areas such as the machine shop, measurement equipment, and computer facilities.

One immediate necessity was the fabrication of a galvanized sheet metal and plywood oil trap and floor protector. Normal operation of the machine is messy

and produces a continual flow of oil to the floor and surrounding work surfaces. Occassional malfunctions can produce a problematic amount of oil in a matter of the seconds it takes to restore control or shut off the system. Connection of the hydraulic lines was straight-forward and quickly accomplished. High pressure hoses were obtained from a local supplier by special order.

Fortunately, we had been forewarned about the impaling capabilities of the machine under certain extrusion conditions specifically, pinch-off or over-extrusion can results in a sudden loss of pressure in the extrusion chamber as the end of the billet is extruded through the die. The next task was therefore to build a suitable safety barrier which could be placed in position to absorb or deflect any projectiles. To handle the design and construction of the safety barrier, we called upon the services of a graduate student on the project. The result of his work is a sturdy wood barrier five feet by eight feet and four inches thick mounted on a two and a half foot tall pedestal with casters on the bottom. Both the forces involved in the operation of the machine and the adequacy of the barrier were dramatically and unexpectedly illustrated when during one of the first extrusions a loud bang was followed immediately by a puff and trail of smoke. As the smoke cleared and the panic subsided, we observed a two foot long aluminum rod quivering dead center in the strategically located safety barrier. Figure 1 shows a photograph of the hydrostatic extrusion machine in its present location. The movable barrier (in the background of the photograph) is in the proper position for the direct mode of hydrostatic extrusion.

The first few trial extrusions were successful. Figure 2 shows three sample extrusions of 1100 aluminum. In each case the original rod diameter was .750". Listing from top to bottom, the extrusion ratios are 16:1, 25:1 and 36:1. Unfortunately, problems quickly developed in the hydraulic extrusion piston and the pressure intensifier. Since both of these failures were unexpected, it seemed appropriate to dismantle and clean the entire machine, check alignment and tolerances, and

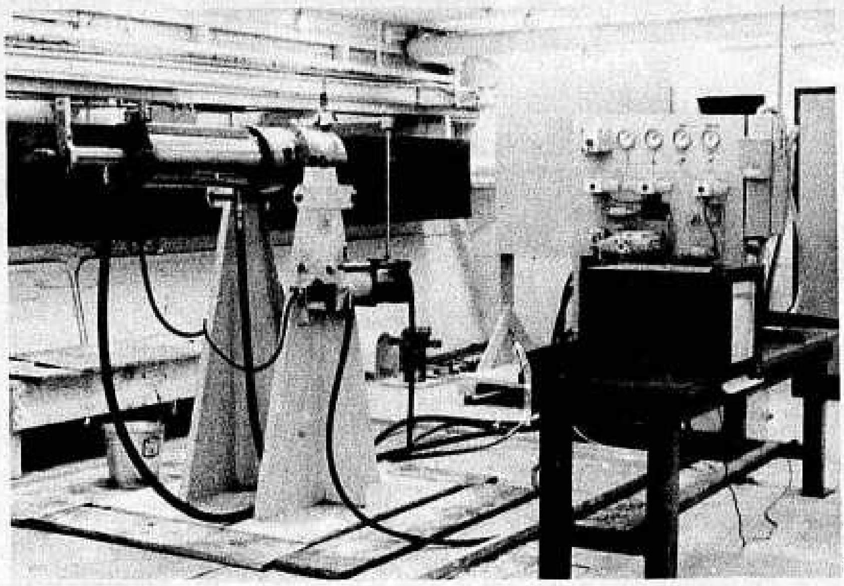

Figure 1. The above photograph shows the hydrostatic extruder in its present location. Both the hydrostatic control panel and protection barrier are visible in the background. The barrier is properly positioned for the direct extrusion mode.

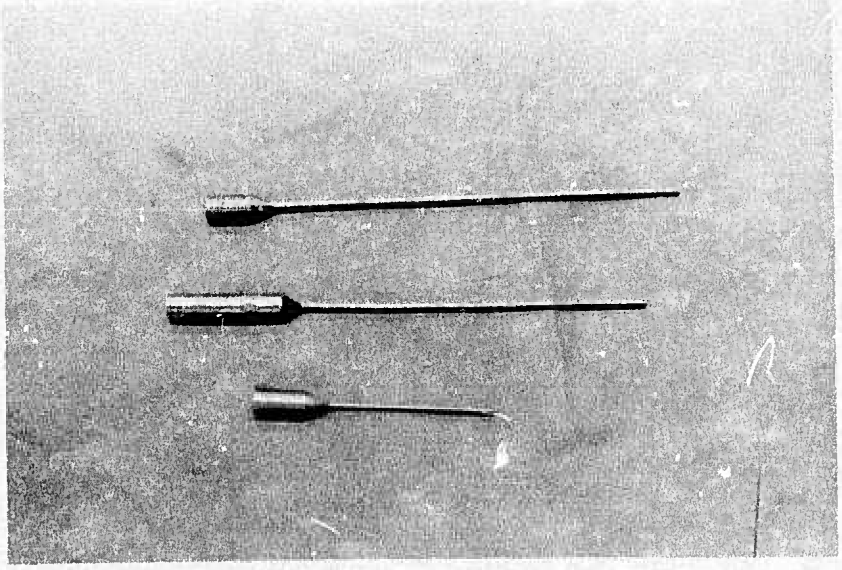

Figure 2. Sample aluminum extrusions. Initial billet diameter was .750". Listing from top to bottom, the extrusion rations are 16:1, 25:1, and 36:1.

replace all seals. The original problem we faced with the hydraulic piston was that the die stem assembly would not retract. When the hydraulic pressure was boosted to increase the force of retraction, the die stem slipped away from the holding collet and oil gushed profusely from both ends of the machine. The problem was diagnosed as jammed die stem guides -- probably the result of bent die stem pins. Special tools were machined and fabricated to facilitate the removal of the cylinder's internal parts. Disassembly did, in fact, reveal bent or scored guide pins. We surmised that either the guides had not been aligned properly originally and that this had caused the pins to bend as the piston was fully retracted or, alternatively, that the piston had simply been over contracted. The defective set of pins was replaced with a new set which was machined at Stanford.

The pressure intensifier was not recharging properly. It was not retracting even under high load. Both the intensifier piston and the cylinder were removed and measured. It was obvious from simply looking at the piston and high pressure seals that after only approximately 200 cycles, the intensifier was showing signs of uneven wear. The dimensions were definitely already beyond the tolerances specified in the blueprints. In an attempt to minimize further delays, it was decided, in consultation with Mr. Keathley at Revere Copper and Brass, that we would try to repair the intensifier at Stanford by first carefully polishing and replacing the worn seals. Both problems were satisfactorily resolved and the machine was returned to working condition. To our dismay, the first attempts to extrude after repairs were made resulted in failure. Typically the machine would either stall at maximum extrusion pressure (250,000 psi) or the material would characteristically pinch off. To complicate matters further the flow control valve was not operating properly in the factory installed position. The idea had been to place the valve in parallel with the extrusion piston so that the meter acted as a bleeder to vent excess oil to the tank (the slower the extrusion

rate, the more venting to tank). However, it became obvious that the venting rate was very pressure sensitive and rate control was impossible. The valve was originally placed in parallel to avoid the loss in maximum attainable extrusion pressure resulting from the pressure drop through the flow meter when it is connected in line. We had no choice but to sacrifice a possible five percent drop in extrusion pressure for rate control. The lines were therefore rerouted so that the valve is now in series with the extrusion piston. Rate, although not yet entirely steady, does appear to be reasonably controllable and pressure independent. This will become clearer as we utilize our monitoring systems.

We are uncertain as to why many extrusion attempts were unsuccessful. Cracked dies, too much friction between die and liner and too high a reduction ratio for the specific extrusion material were all suspected. After several failures, close examination of the dies supplied by Revere Copper and Brass revealed longitudinal cracks. Whether these cracks created the stall outs or were caused by the high static pressure is not certain. However, cracked dies can open up under pressure and lead to a loss of lubricant (that is, loss of hydrodynamic lubrication), the consequence being a sharp increase in the applied load with an actual loss in hydrostatic pressure. Increased friction between liner and die would have the same effect; that is, the applied load would need to increase in proportion to a frictional increase in order for extrusion to take place. If the applied load necessary to overcome the wall friction is beyond the machine's capability, the machine stalls. For this reason die diameter and work chamber liner dimensions are checked frequently. Normally when under pressure the die should be oversized by .0005" to .0010". This provides the metal to metal contact for hydrostatic medium retention, but still permits most of the applied force to be transmitted to the extrusion sample via the hydrostatic fluid. Friction buildup was definitely a problem for us since the liner became badly scored. This necessitated the removal of the liner and rehoning to a

new inside dimension. Larger dies would have to be machined to fit or old ones copper plated -- a practice which should yield several extrusions per coating.

In any event, it was obvious that we would have an almost immediate need to design and order additional dies which would enable us to extrude to a greater variety of diameters. We decided to use the Stanford machine shop facilities and avoid going to outside contractors for this work. Thus we could maintain greater control and at the same time retain flexibility for modifications found desirable. Once initial decisions as to materials and optimum heat treatments were made, die manufacture became routine. In recent weeks our time has been occupied with problems inherent in monitoring the several extrusion variables. It was noted in an earlier report that an automatic system to monitor die jacket pressure was essential for the prevention of internal damage to the extrusion chamber. The ultra high pressure developed in the extrusion chamber can only be contained by a corresponding maintenance of high pressure in the die jacket. If die jacket pressure were to drop the load chamber liner would likely split. We conceived a plan which calls for mounting strain gages externally on the high pressure cylinder and would thus correlate cylinder strain to die jacket pressure. When a specific lower or upper limit is exceeded the machine should shut down automatically. This safety system has been designed, but not built.

For our initial studies the most important problem facing us is the accurate determination of the extrusion temperature. We would like to know if extrusion conditions are isothermal; and if not, how much heating takes place. Much thought and energy went into devising various schemes for measuring temperature. The object was to get an accurate reading at the die exit where the temperature should be the greatest. Access to the die was a problem since space is extremely limited. We were hoping for rather high response times in order to monitor possible temperature fluctuations during stick-slip or strain rate changes (that is increase in extrusion

rate during test). Methods such as infra-red sensors do not appear feasible at the moment due to inaccessibility of the die. The tentative decision was to design and build a prototype model using thin, unshielded thermocouples (see Figure 3). The thermocouple was surrounded by a thermal insulating ring of teflon to prohibit any thermal contact between sensor and machine. Clearance between sensor and extruded rod was to be no more than .005". This close tolerance was chosen so that the sensor would be in contact with either the extruded rod or the thin film of hydrostatic medium which is also extruded as a coating. Since monitoring is considerably easier in the direct mode of extrusion where the die is stationary our initial measuring attempts concentrated on this mode. The appropriate plugs and dies were machined and heat treated and the machine set up to operate in the direct extrusion mode. Preliminary tests demonstrate that the thermocouple setup does work. However, there is still some fluctuation in the temperature. This is probably not the true response of the extruded material. More likely, it is the result of the thermocouple making and breaking contact with the film or extruded rod. In addition, the response time must be calibrated; and most importantly the accuracy of our temperature measurements must be determined.

Although the extruder will be used mainly as a processing tool, we want to develop the capabilities to record the history of each extrusion. Of course, knowledge of the temperature of extrusion, extrusion pressure, and rate will be important in determining the mechanical properties of these materials. But this information will also be useful in analyzing the extrusion process, die efficiency, lubricant efficiency, or other extrusion phenomena.

We have purchased a linear potentiometer to be used as a displacement gage to position accurately the extrusion piston versus time during an extrusion. In the same manner the hydrostatic pressure and extrusion temperature will be monitored with respect to time. Thus a record of the metal working history of each point on

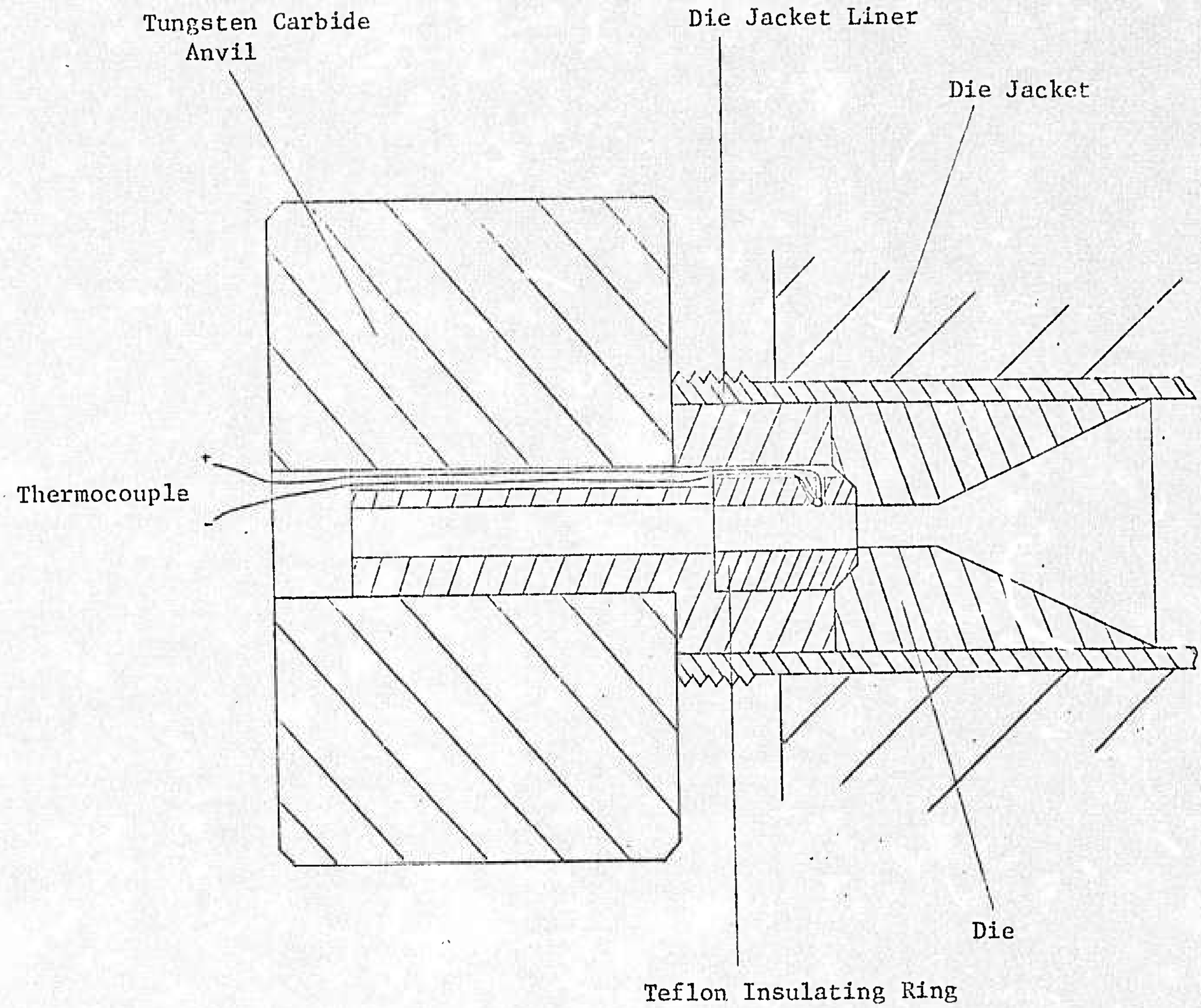

Figure 3. Thermocouple design for measuring the temperature of extruded products at the die exit.

the extruded rod will be preserved for later reference. A Hewlett-Packard computer system with analog to digital conversion, data processing capabilities, plotters and visual display has recently been installed in the Structures Laboratory and has been made available for our use. We will be plugging our strain gages and sensors into it directly for data storage and manipulation.

Our immediate future will be concerned with integration of the Hewlett-Packard computer into our extrusion system. Three weeks to one month will probably be required to set up the equipment and learn the techniques for computer operation and commands.

Simultaneous with efforts to implement the computer analysis will be trial extrusions of harder materials. Our goal is to develop capabilities for warm and hot extrusions. Battelle reportedly has the ability to extrude at 1000°C with pressures to 250,000 psi. Unfortunately, details of Battelle's technique and equipment are proprietary. Our hydrostatic extrusion machine was not designed specifically with hot extrusion in mind but we hope that ideas we have developed will permit extrusion approaching 1000°C and at pressures of 250,000 psi. Barring major unforseen difficulties, extrusion temperatures of 500-600°C at full pressure (250,000 psi) should be readily achievable. The first precaution must be to insure against overheating of the hydraulic oil and loss of tempered strength of the die jackets. At higher temperatures, heat retention, thermal shock, thermal expansion and selection of material, hydrostatic medium and lubricant are all factors which may be problematic.

We considered both internal and external heating of the billet. The alternative to heat externally and quickly transfer billet and hydrostatic medium offered the best chance for immediate success. Therefore, our efforts have concentrated on perfecting this system. Figure 4 illustrates the proposed modification of the present system to accomodate warm extrusion (200-750°C). We are presently still designing, checking into the materials selection and also researching other related work which has recently been published.

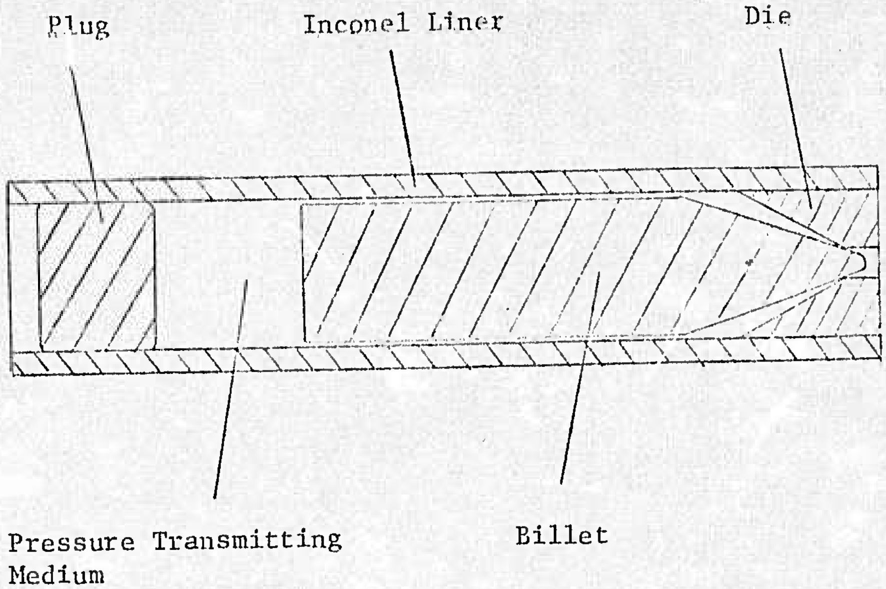

Figure 4a. Preheating container for warm extrusion.

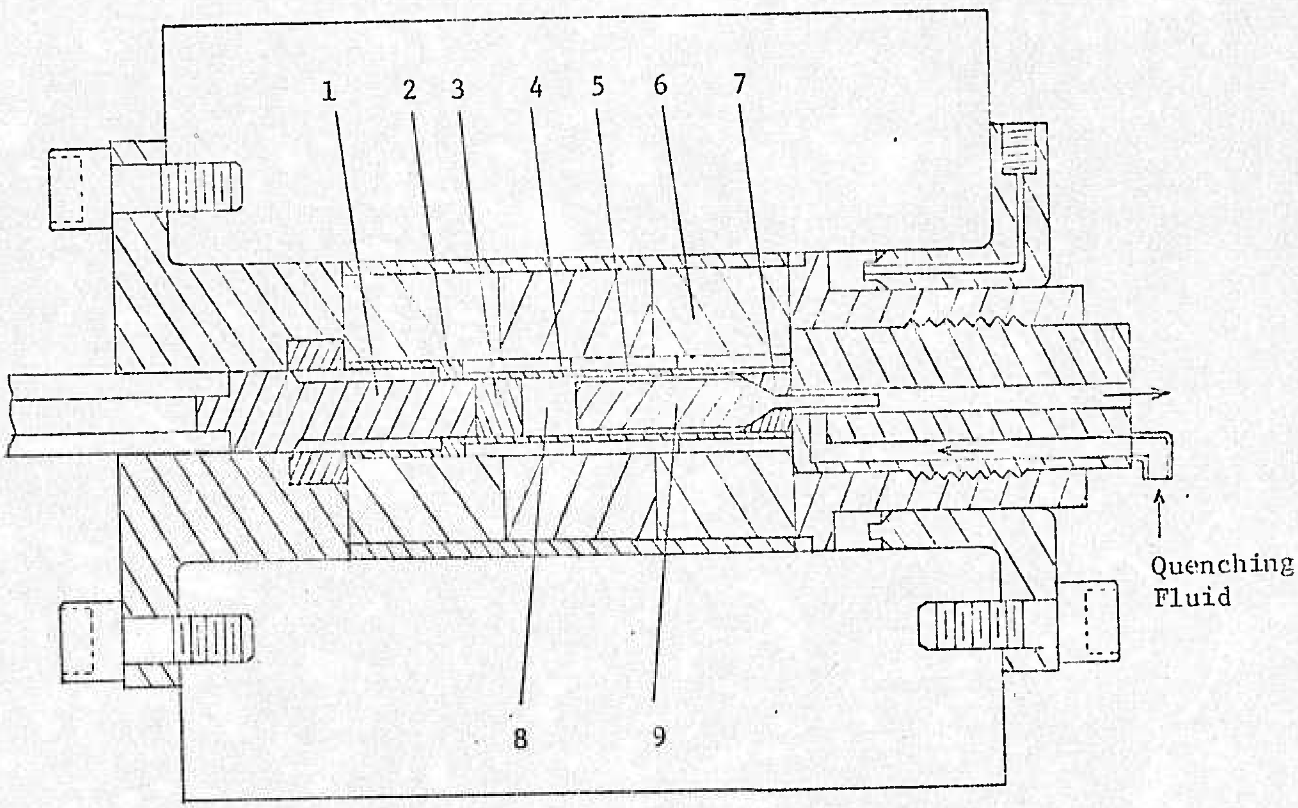

Figure 4b. Extrusion chamber modified for warm extrusion and quenching. (1) Tool steel plunger (2) Inconel sleeve (3) Plug (4) Insulating sleeves (5) Inconel Liner (6) Die jacket (7) Die (8) Pressure transmitting medium and lubricant (9) Billet.

II. The Basis for an Elastic-Plastic Code
(Professor Erastus H. Lee)

Introduction

For satisfactory analytical assessment of metal forming problems, it is necessary to evaluate the varying stress distribution in an element since forming defects, such as the development of internal cracks, can depend on stress history, and residual stresses can be important in deciding the utilization of a formed part. Elastic characteristics play an important role in the determination of stress, even in combination with extensive plastic flow which may involve strains a thousand times elastic strain magnitudes. Thus analysis of metal forming problems for such assessments must be based on elastic-plastic theory. The same is true for other stress analysis problems when plastic flow occurs.

Because the plasticity laws are incremental in nature, they result in relations between stress-rate and strain-rate, or equivalently in numerical evaluations, between stress and strain increments. For the commonly adequate rate-independent laws, linear relations between stress-rate and strain-rate arise, the coefficients being functions of the current stress for the common laws when plastic flow is taking place, and otherwise the elastic laws apply in incremental form. Because of the structure of these laws, elastic-plastic problems are commonly solved in terms of equations for stress-rates and strain-rates, containing stresses as coefficients. Thus a time step forward, Δt, from the current situation at time t gives the solution at time $t + \Delta t$ with stress $\underset{\sim}{\sigma}(t) + \dot{\underset{\sim}{\sigma}}\Delta t$, and similarly for other variables. Then a new time step can be taken and the process repeated. $\overset{\circ}{\underset{\sim}{\sigma}}$ is the appropriate stress-rate.

Extensive studies of the application of these laws to stability and uniqueness of solutions have been made by Hill (see, for example [1], [2], [3]*) where he shows that care in the selection of stress definitions and stress-rate and strain-rate expressions is important for a satisfactory development of the theory. Rice [4] has pointed out that such questions are also important in developing a satisfactory theoretical basis for elastic-plastic stress analysis, particularly in the common circumstance that the tangent modulus in plastic flow is of the order of the stress. Convected and rotation terms then become important in the stress-rate expression, and analogously stress variables should be selected so that the influence of rate of deformation of the boundaries of the body does not affect the variational principle which replaces the equilibrium equation. This requirement can be achieved by using the unsymmetric nominal stress (Piola-Kirchoff I) in which the stress is defined as force per unit undeformed area. The variational principle then involves an integral over the undeformed body which is fixed.

Plastic flow is essentially a fluid type phenomenon which can be most conveniently expressed in terms of the current configuration of the material. Thus a reference configuration which remains invariant throughtout the motion is not appropriate and so the current configuration is adopted as the reference state for evaluation of the deformation from t to $t + \Delta t$, where Δt is sufficiently small for adequacy of first order theory.

The framework described above provides a satisfactory foundation for a finite-element elastic-plastic code as discussed by McMeeking and Rice [5]. In effect, by choosing the current configuration as the reference state the

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*Numbers in square brackets refer to the bibliography.

Cauchy stress (or true stress in Cartesian coordinates) the unsymmetric nominal stress (Lagrange or Piola-Kirchhoff I) and the symmetric nominal stress (Kirchhoff or Piola-Kirchoff II) all have identical values at the current time which facilitates utilization of the appropriate stress for the appropriate component of the calculation. Although the stress components themselves are identical, rates of change of the different stresses are not the same.

Development of the Theory

Following Hill [2] and using for the most part his notation, we consider the unsymmetric nominal stress (variously referred to as Lagrange or Piola-Kirchoff I) s_{ij} defined so that the jth component of the force transmitted across a deformed element, which in the initial or reference state had area $d\overset{\circ}{S}$ and unit normal $\overset{\circ}{\nu}_i$, is

$$d\overset{\circ}{S}\,\overset{\circ}{\nu}_i\,s_{ij} = dF_j \tag{1}$$

Hill considers (p. 214 of [2]) rate or flow type constitutive laws of the type

$$\dot{s}_{ij} = \frac{\partial E}{\partial(\partial v_j/\partial X_i)} \tag{2}$$

where X_i are rectangular Cartesian coordinates in the initial or reference configuration, E is a homogeneous function of degree two in the velocity gradients, $\partial v_j/\partial X_i$, and where $\dot{s}_{ij}$ is the partial time derivative at fixed $\underset{\sim}{X}$, i.e. a material derivative at a particle. The velocity $v_j(\underset{\sim}{X},t)$ gives the distribution at time t expressed in the initial coordinates of the corresponding material points (note that a tilde under a symbol denotes vector or tensor in absolute notation).

We consider boundary value problems in which for a volume $\overset{\circ}{V}$ in the reference state, at time t stress rates $\dot{s}_{ij}(\underset{\sim}{X},t)$ and $v_i(\underset{\sim}{X},t)$ are sought for prescribed nominal traction rates $\dot{F}_j$ over the part of the surface $\overset{\circ}{S}_F$, velocity v_j over the remainder of the surface $\overset{\circ}{S}_v$ and body force rates $\dot{g}_j$ per unit initial volume. Then the variational principle

$$\delta\left[\int_{\overset{\circ}{V}} E\, d\overset{\circ}{V} - \int_{\overset{\circ}{S}_F} \dot{F}_j v_j d\overset{\circ}{S} - \int_{\overset{\circ}{V}} \dot{g}_j v_j d\overset{\circ}{V}\right] = 0 \tag{3}$$

in the class of continuous differentiable velocity fields satisfying the velocity boundary condition on $\overset{\circ}{S}_v$, characterizes the solution, for it yields the equilibrium equations

$$\frac{\partial \dot{s}_{ij}}{\partial X_i} + \dot{g}_j = 0 \tag{4}$$

and the boundary traction rate condition

$$\overset{\circ}{\nu}_i \dot{s}_{ij} = \dot{F}_j \tag{5}$$

for nominal stress, s_{ij}, and the reference geometry.

In writing the elastic-plastic constitutive relation, we wish to associate the velocity gradient in (2) with the rate of deformation or velocity strain:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{6}$$

where x_i are Cartesian coordinates expressing the position of

particles in the deformed body

$$x_i = x_i(\underset{\sim}{X}, t) \tag{7}$$

Thus to permit simultaneous use of (3) and the plasticity laws expressed in the usual form of rate of deformation of the current configuration, Hill takes the current configuration to be the reference state, and hence

$$X_i = x_i(\underset{\sim}{X}, t) \tag{8}$$

for a particular t. The theory is expressed in this form for evaluation of $\dot{s}_{ij}$ and v_j and hence the solution and configuration at $t + \Delta t$, which provides a new reference state for evaluation of the next time step Δt.

Note that at the instant t, when the current and reference configurations are identical, the nominal stress components s_{ij} are equal to the Cauchy or true stress components σ_{ij}, so that at this instant s_{ij} is symmetric.

The device of selecting the configuration at time t to be the reference state for evaluation of the solution at time $t + \Delta t$ thus permits simultaneous use of the convenient variational principle (3) in terms of nominal stress s_{ij} and a fixed geometry and the familiar plasticity laws expressed in terms of the Cauchy, or true stress, σ_{ij}.

By working with curvilinear convected coordinates, ξ^α, having an arbitrary configuration in the reference or initial state, Hill([2], p. 219 ff.) shows that the rate potential (2) generates associated rate potentials for other stresses and stress rate expressions, some of which are more convenient for representing elastic plastic laws.

Consider curvilinear coordinates ξ^{α} in the initial or reference state with base vectors $\overset{\circ}{g}_{\alpha}$. After deformation as shown in Fig. 1 these become $\xi^{\alpha'}$, having the same values for the same material particles as ξ^{α}. Then $\xi^{\alpha'}$ are the convected coordinates and the corresponding base vectors are $\underset{\sim}{g}_{\alpha'}$, which are $\underset{\sim}{\overset{\circ}{g}}_{\alpha}$ deformed by the motion. Then the Cauchy stress tensor $\underset{\sim}{\sigma}$ has contravariant components $\sigma^{\alpha'\beta'}$ in convected coordinates such that the force $\underset{\sim}{dF}$ transmitted across an element of the deformed body of area dS and unit normal vector $\nu_{\alpha'}$ is given by:

$$\underset{\sim}{dF} = \sigma^{\alpha'\beta'}(\nu_{\alpha'}dS)\underset{\sim}{g}_{\beta'} \tag{9}$$

Since the primed coordinates are made evident by the notation ν and $\underset{\sim}{g}$ for normal and base vectors in the deformed state, the primes will usually be dropped hereafter, for $\xi^{\alpha} = \xi^{\alpha'}$ denote the same material point.

For the Lagrange or Piola-Kirchhoff I stress, s_{ij}, the force computed in the reference frame is that actually acting across the deformed element (see Fung [6] p. 437 for the usual definition in Cartesian coordinates). For the present consideration of convected coordinates, the expression for $\underset{\sim}{dF}$ thus takes the form

$$\underset{\sim}{dF} = s^{\alpha\beta}(\overset{\circ}{\nu}_{\alpha}\, d\overset{\circ}{S})\underset{\sim}{\overset{\circ}{g}}_{\beta} \tag{10}$$

For the Kirchhoff or Piola-Kirchhoff II stress, τ_{ij}, Fung points out that the force vector computed in the reference frame must be transformed by the motion to give the actual force across the deformed element, so that for convected coordinates

$$d\underset{\sim}{\overset{\circ}{F}} = \tau^{\alpha\beta}(\overset{\circ}{\nu}_{\alpha}d\overset{\circ}{S})\underset{\sim}{\overset{\circ}{g}}_{\beta}$$

becomes

$$d\underset{\sim}{F} = \tau^{\alpha\beta}(\overset{\circ}{\nu}_\alpha d\overset{\circ}{S})\underset{\sim}{g}_\beta \tag{11}$$

Nanson's relation for area element:

$$\rho\nu_\alpha dS = \overset{\circ}{\rho}\overset{\circ}{\nu}_\alpha d\overset{\circ}{S} \tag{12}$$

then gives, using (9) and (11)

$$\tau^{\alpha\beta} = \frac{\overset{\circ}{\rho}}{\rho}\sigma^{\alpha\beta} = J\,\sigma^{\alpha\beta} \tag{13}$$

where J is the Jacobian of the transformation from the reference to the deformed state, i.e.

$$\underset{\sim}{g}_\alpha \cdot \underset{\sim}{g}_\beta \times \underset{\sim}{g}_\gamma = J(\overset{\circ}{\underset{\sim}{g}}_\alpha \cdot \overset{\circ}{\underset{\sim}{g}}_\beta \times \overset{\circ}{g}_\gamma) \tag{14}$$

Fig. 2 shows the particular situation when the ξ^α coordinates in the reference frame are Cartesian, X^α. The Cartesian coordinates representing points in the deformed body according to the point transformation, equ. (7), are x^i. The usual definition of the Kirchhoff or Piola-Kirchhoff II stress (see Fung [6] p. 439) is given in terms of this point transformation. From equ. (7), define the deformation gradient

$$\underset{\sim}{F} = \frac{\partial x^i}{\partial X^\alpha} \tag{15}$$

Then the Piola-Kirchhoff II stress $\underset{\sim}{\tau}$ is given in terms of the Cauchy stress in Cartesian coordinates $\underset{\sim}{x}$, $\overset{c}{\underset{\sim}{\sigma}}$ by

$$\underset{\sim}{\tau} = J\,\underset{\sim}{F}^{-1}\,\overset{c}{\underset{\sim}{\sigma}}\,\underset{\sim}{F}^{-1^T} \quad ([7]\text{ p. }125) \tag{16}$$

where $J = \det(\underset{\sim}{F})$. This is in accord with (13) where $\sigma^{\alpha\beta}$ are the contravariant components of the Cauchy stress with respect to the

convected coordinates $X^{\alpha'}$; for if $\overset{c}{\sigma}^{ij}$ are the Cartesian coordinates of $\underset{\sim}{\sigma}$ with respect to $\underset{\sim}{x}$, then the tensor change of variables law gives

$$\sigma^{\alpha'\beta'} = \frac{\partial X^{\alpha'}}{\partial x^i} \frac{\partial X^{\beta'}}{\partial x^j} \overset{c}{\sigma}^{ij} \tag{17}$$

in terms of the coordinate transformation from $\underset{\sim}{x} \rightarrow \underset{\sim}{X}'$ in the deformed geometry. Because of the property of convected coordinates that $X^{\alpha} = X^{\alpha'}$ for the same particle, (7) also expresses the coordinate transformation, and (13) and (17) are seen to be equivalent to (16). Note that $s^{\alpha\beta}$ and $\tau^{\alpha\beta}$ are tensor densities and not absolute tensors, so that equations such as (2) are not pure tensor relations

For rate independent constitutive relations the rate potential function E in (2) is a homogeneous function of degree two. For the choice that the reference state is the current state, (2) follows from the existence of an associated rate potential function $F(\varepsilon_{\alpha\beta})$, of the strain-rate components

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) \tag{18}$$

where $\underset{\sim}{v}$ is the velocity vector in the deforming body and the comma denotes covariant differentiation. This function generates $\dot{\tau}^{\alpha\beta}$ through

$$\dot{\tau}^{\alpha\beta} = \frac{\partial F}{\partial \varepsilon_{\alpha\beta}} \tag{19}$$

where the superposed dot indicates time derivative of the convected components or convected derivative. This is equivalent to the partial time derivative at fixed $\underset{\sim}{X}$, or material derivative. The structure of (19) indicates that since $\underset{\sim}{\tau}$ is a tensor density, F is a scalar density and

not an absolute scalar invariant. Some details of the derivations needed are given in the Appendix, in particular for the relationship between $\dot{s}^{\alpha\beta}$ and $\dot{\tau}^{\alpha\beta}$ which for the particular choice of reference state mentioned takes the form

$$\dot{s}^{\alpha\beta} = \dot{\tau}^{\alpha\beta} + \tau^{\alpha\gamma} v^{\beta}{}_{,\gamma} \tag{20}$$

Now from (19) and (20)

$$\dot{s}^{\alpha\beta} = \partial F/\partial(\varepsilon_{\alpha\beta}) + \tau^{\alpha\beta} v^{\beta}{}_{,\gamma}$$

multiplying both sides by $v_{\beta,\alpha}$ gives

$$\dot{s}^{\alpha\beta} v_{\beta,\alpha} = [\partial F/\partial(\varepsilon_{\alpha\beta})]\varepsilon_{\alpha\beta} + \tau^{\alpha\gamma} v^{\beta}{}_{,\gamma} v_{\beta,\alpha} \tag{21}$$

using (18) and the fact that $\tau^{\alpha\beta}$ is symmetric. Now $F(\varepsilon_{\alpha\beta})$ is also a homogeneous function of degree two since for plasticity (19) is rate independent, hence Euler's theorem for homogeneous functions permits us to write (21) in the form

$$\dot{s}^{\alpha\beta} v_{\beta,\alpha} = 2F(\varepsilon_{\alpha\beta}) + \tau^{\alpha\gamma} v^{\beta}{}_{,\gamma} v_{\beta,\alpha} \tag{22}$$

Again using Euler's theorem, this is consistent with (2) in terms of convected coordinates

$$\dot{s}^{\alpha\beta} = \partial E/\partial(v_{\beta,\alpha})$$

and

$$2E(v_{\beta,\alpha}) = 2F(\varepsilon_{\alpha\beta}) + \tau^{\alpha\gamma} v^{\beta}{}_{,\gamma} v_{\beta,\alpha} \tag{23}$$

With $\tau^{\alpha\gamma}$ known, this establishes that E is a homogeneous function of second degree in $v_{\beta,\alpha}$.

The laws of plasticity are normally obtained by measuring "true stress" and increments of strain defined in terms of Cartesian coordinates in the current configuration without rotations occurring. Since the theory must apply in the presence of rotations, their influence must not affect the stress rate term in the constitutive relation, hence a Jaumann or spin-invariant stress rate seems appropriate. We will work in terms of the configuration at time t, which is the reference eonfiguration, and utilize Cartesian coordinates $\underset{\sim}{x}$.

In formulating the finite element theory for numerical analysis of elastic-plastic problems, we wish to use the variational relation (3) in terms of $\overset{\circ}{s}_{ij}$ and a constitutive relation associated with (19) in terms of τ_{ij} since it can be conveniently associated with measurements of plasticity laws. We have seen that a law in the form (19) implies the validity of (2) and hence the variational principle. As pointed out in [5] this strueture in terms of τ_{ij} leads to symmetric stiffness matrices in the finite element formulation whieh simplifies the numerical procedures.

Now the relationship between the nominal stress $\underset{\sim}{s}$ and the Cartesian true stress $\underset{\sim}{\overset{c}{\sigma}}$ is

$$\underset{\sim}{F}\,\underset{\sim}{s} = J\,\underset{\sim}{\overset{c}{\sigma}} \tag{24}$$

see, for example [7], p. 125, where the nominal stress is defined as the transpose of $\underset{\sim}{s}$ (or it ean be deduced from (9), (10) and (17)). Taking the material derivative of (24), and noting that F and J are unity for coineident reference and current configurations, one obtains

$$\overset{\circ}{s}_{ij} + s_{kj}\frac{\partial v_i}{\partial x_k} = \overset{c}{\sigma}_{ij}\frac{\partial v_k}{\partial x_k} + \overset{\circ}{\overset{c}{\sigma}}_{ij} \tag{25}$$

The difference between the Jaumann derivative of $\overset{c}{\underset{\sim}{\sigma}}$ and its material derivative is the contribution of the rotation of the axes which rotate with the body according to the anti-symmetric tensor

$$\frac{1}{2} \left.\frac{\partial v_i}{\partial x_j}\right|_A \tag{26}$$

where A denotes the anti-symmetric part. Using $\mathcal{D}/\mathcal{D}t$ to denote the Jaumann derivative, this determines (see Prager [8] p. 155)

$$\frac{\mathcal{D}\overset{c}{\sigma}_{ij}}{\mathcal{D}t} - \dot{\overset{c}{\sigma}}_{ij} = - \sigma_{ik} \left.\frac{\partial v_j}{\partial x_k}\right|_A - \sigma_{kj} \left.\frac{\partial v_i}{\partial x_k}\right|_A \tag{27}$$

Combination of (25) , (27) and (6) gives

$$\dot{s}_{ij} = \left(\frac{\mathcal{D}\overset{c}{\sigma}_{ij}}{\mathcal{D}t} + \overset{c}{\sigma}_{ij} \frac{\partial v_k}{\partial x_k}\right) - (\sigma_{ik} D_{jk} + \sigma_{kj} D_{ik}) + \sigma_{ik} \frac{\partial v_j}{\partial x_k} \tag{28}$$

Strictly speaking, τ_{ij} only defines the Kirchhoff stress for convected coordinates. Being a tensor density, definition for other coordinates does not follow the tensor relation (17) which yields other components of the stress tensor $\underset{\sim}{\sigma}$. However, we wish to define the first term on the right hand side of (28), and this can be written $\mathcal{D}\overset{c}{\tau}_{ij}/\mathcal{D}t$ where $\overset{c}{\tau}_{ij} = J \overset{c}{\sigma}_{ij}$. This result follows by taking the Jaumann derivative of the product and noting that the derivative of the scalar density J is unaffected by the rotation of the axes and that the instantaneous value of J is unity. Note that to define a Jaumann derivative of $\underset{\sim}{\tau}$, $\underset{\sim}{\tau}$ must be defined for other than convected coordinates since rotated rigid coordinates are involved. Thus (28) can be written

$$\dot{s}_{ij} = \frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} - (\sigma_{ik} D_{jk} + \sigma_{kj} D_{ik}) + \sigma_{ik} \frac{\partial v_j}{\partial x_k} \tag{29}$$

Now combining (20) and (29)

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} = \dot{\tau}_{ij} + \sigma_{ik} D_{jk} + \sigma_{kj} D_{ik} \tag{30}$$

In view of (19) written for initial Cartesian coordinates, with $F(D_{ij})$ homogeneous of second order in the strain rates, multiplication of (30) by D_{ij} and using Euler's theorem gives

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} D_{ij} = 2F + \sigma_{ij} D_{jk} D_{ij} + \sigma_{kj} D_{ik} D_{ij}$$

$$= 2F + 2\sigma_{ik} D_{ij} D_{jk} \tag{31}$$

Thus defining

$$G = F + \sigma_{ik} D_{ij} D_{jk} \tag{32}$$

and using Euler's theorem, a third rate potential function is generated:

$$\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t} = \frac{\partial G}{\partial D_{ij}} \tag{33}$$

The classical elastic-plastic isotropically work hardening law (Prandtl-Reuss), commonly giving strain-rate as a linear function of stress-rate, can be inverted to give stress-rate and takes the form ([5] p. 606)

$$\frac{\mathfrak{D}\overset{c}{\sigma}_{ij}}{\mathfrak{D}t} = \frac{E}{1+\nu} [\delta_{ik}\delta_{jl} + \frac{\nu}{1-2\nu} \delta_{ij}\delta_{kl} - \frac{3\overset{c}{\sigma}{}'_{ij}\overset{c}{\sigma}{}'_{kl} \left(\frac{E}{1+\nu} \right)}{2\bar{\sigma}^2 \left(\frac{2}{3}h + \frac{E}{1+\nu} \right)}] \varepsilon_{kl} \tag{34}$$

where E and ν are Young's modulus and Poisson's Ratio, σ' denotes

stress deviator, $\bar{\sigma}$ is the current tensile yield stress and h the current gradient of the true-stress logarithmic plastic strain curve in a tension test. The Jaumann derivative is used for stress-rate as mentioned above in order to eliminate rotation effects, and the last term in the brackets is dropped when the increment of deformation is elastic. We have shown that there is a rate-potential function G for $\overset{c}{\tau}_{ij}$, but one does not exist for $\overset{c}{\sigma}_{ij}$. This means [5] that a non-symmetric finite-element stiffness matrix would be deduced using (34). Replacing $\overset{c}{\sigma}_{ij}$ by $\overset{c}{\tau}_{ij}$ in the stress rate term would yield a symmetric stiffness matrix which would simplify the numerical analysis. Moreover such a change is appropriate in terms of its representation of the physical laws. The J term in

$$\overset{c}{\tau}_{ij} = J\,\overset{c}{\sigma}_{ij} \tag{35}$$

arises in more accurate representation of the laws of elasticity than Hooke's law used in classical elasticity. It is associated with geometrical non-linearity which expresses the non-linear influence of finite strain. In effect it expresses the fact that energy density should be referred to a unit initial volume, for a unit current volume implies change of energy density simply because the amount of material containing it changes. For example, the term (ρ/ρ_0) appearing in equation (26) of [9] is equivalent to replacing $\underset{\sim}{\sigma}$ by $J\underset{\sim}{\sigma}$, and it was pointed out in that paper (p. 935) that such a term provides a good approximation to non-linear elasticity of metals with input of only the two classical elastic constants. A similar modification of the classical laws of plasticity was suggested in [10] where it was found compelling to express the yield stress in terms of $J\underset{\sim}{\sigma}$ (equs. (33) and (34) of [10]). Again this was based on the requirements

of geometrical non-linearity, which of course are independent of specific material characteristics.

Introduction of (2) into the variational principle (3) expresses it in the form

$$\int_{\overset{\circ}{V}} \dot{s}_{ij}\ \delta\left(\frac{\partial v_j}{\partial x_i}\right)\, d\overset{\circ}{V} - \int_{\overset{\circ}{S}_F} \dot{\overset{\circ}{F}}_j v_j\, d\overset{\circ}{S} - \int_{\overset{\circ}{V}} \dot{\overset{\circ}{g}}_j v_j\, d\overset{\circ}{V} = 0 \qquad (36)$$

which, since the reference state is the current state ($\underset{\sim}{X} = \underset{\sim}{x}$) can be written

$$\int_V \dot{s}_{ij}\ \delta\left(\frac{\partial v_j}{\partial x_i}\right) dV - \int_{S_F} \dot{F}_j v_j\, dS - \int_V \dot{g}_j v_j\, dV = 0 \qquad (37)$$

Substitution of (29) and applying algebraic manipulation based on symmetries then yields the variational principle in the form (see equ. (5) of [5])

$$\int_V \left[\frac{\mathfrak{D}\overset{c}{\tau}_{ij}}{\mathfrak{D}t}\ \delta(D_{ij}) - \frac{1}{2}\,\sigma_{ij}\,\delta(2D_{ik}D_{kj} - v_{k,i}\,v_{k,j}\right]dV$$

$$- \int_{S_F} \dot{F}_j v_j\, dS - \int_V \dot{g}_j v_j\, dV = 0 \qquad (38)$$

where the subscript $,i$ denotes the operation $\partial/\partial x_i$. Now utilizing (33), the principle takes the form (see equ. (15) of [5]).

$$\delta\left[\int_V G(D)\ dV - \frac{1}{2}\int_V \sigma_{ij}(2D_{ik}D_{kj} - v_{k,i}\,v_{k,j})dV\right.$$

$$- \int_{S_F} \dot{F}_j v_j\, dS - \int_V \dot{g}_j v_j\, dV = 0 \qquad (39)$$

By (33), and using Euler's theorem

$$\frac{\mathcal{D}\overset{c}{\tau}_{ij}}{\mathcal{D}t} = \frac{\partial G}{\partial D_{ij}} = \frac{\partial^2 G}{\partial D_{ij}\partial D_{kl}} D_{kl} = \mathcal{L}_{ijkl} D_{kl} \tag{40}$$

since $\partial G/\partial D_{ij}$ is homogeneous of first degree in $\underset{\sim}{D}$, hence $\mathcal{L}_{ijkl}$ is symmetric in $ij \leftrightarrow kl$ as well as $i \leftrightarrow j$ and $k \leftrightarrow l$. Since G is homogeneous of second degree

$$G(\underset{\sim}{D}) = \frac{1}{2} \frac{\mathcal{D}\overset{c}{\tau}_{ij}}{\mathcal{D}t} D_{ij} \tag{41}$$

The symmetry implicit in the variational principles (38) and (39) with (40) and (41) imply symmetric stiffness matrices which carry through to the finite-element formulation [5].

Discussion

As discussed by Rice [4] and McMeeking and Rice [5], the development reviewed in these notes brings out the importance of convection effects and appropriate stress-rate definitions in formulating elastic-plastic theory. Thus the precise analysis of continuum theory must be applied to obtain reliable results. Common small strain assumptions are not valid even for incremental theory based on small strain increments. This situation is implicit in comparison of the first and second terms in the first volume integral in the variational principle (38). In plastic flow the coefficient of $\delta\underset{\sim}{D}$ in the first term is $\sim h\underset{\sim}{D}$, where h is the gradient of the tensile stress-plastic strain curve, whereas the second term is $\sim \underset{\sim}{\sigma}\underset{\sim}{D}\delta\underset{\sim}{D}$. The second term and the difference between the Jaumann derivative and other simpler time derivatives can only be neglected if $h >> \underset{\sim}{\sigma}$.

The relative error in neglecting such terms is effectively independent of the magnitude of the strain increment adopted, so that small steps do not permit simplification in this regard. For many metals $h \sim \sigma$.

It is interesting to note that the complications which arise in elastic-plastic analysis result from the elastic component of strain, and not the plastic. For stress and strain deviators, elastic-ideally plastic deformation with a Mises yield condition, $J_2 = \sigma'_{ij}\sigma'_{ij}/2 = k^2$, satisfies

$$D'_{ij} = \overset{\circ}{\sigma}'_{ij}/2G + \dot{\lambda}\sigma'_{ij} \tag{42}$$

where λ is a parameter. Multiplication of (42) by σ'_{ij} gives

$$\dot{\lambda} = \sigma'_{ij}D'_{ij}/2J_2 \tag{43}$$

For isotropic work hardening with a Mises yield condition, (42) takes the form

$$D'_{ij} = \overset{\circ}{\sigma}'_{ij}/2G + f(J_2)\dot{J}_2\sigma'_{ij} \tag{44}$$

The first terms on the right hand sides of (42) and (44) are the elastic strain rate components. Thus rigid-ideally plastic theory ((42) with the $\overset{\circ}{\underset{\sim}{\sigma}}$ term deleted) gives a relation between stress and velocity gradient with no complications due to stress rate, which greatly simplifies the analysis, apart from the difficulty of determining the rigid regions. Similarly for work hardening rigid-plastic analysis ((44) with the $\overset{\circ}{\underset{\sim}{\sigma}}$ term deleted), only the rate of change of a stress invariant occurs, which is simpler to include than a tensor rate. For elastic-plastic theory it is the elastic term which introduces stress-rate and the consequent complications.

Many technologically important metal-forming problems are steady state processes in which $\partial\sigma_{ij}/\partial t|_{\underset{\sim}{x}} = 0$. In planning to use an analysis of the type considered here to evaluate such situations, it is fortunate that the stress-rate terms appearing in the variational principle, $\dot{s}_1$, or $\mathcal{D}\overset{c}{\tau}_{ij}/\mathcal{D}t$, do not approach zero in the steady case, hence singular computational conditions need not be anticipated. This is not the case in some simpler and inadequate approaches to this problem in which sufficient care was not devoted to the appropriate choice of stress-rate definition.

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References

1. R. Hill. A general theory of uniqueness and stability of elastic-plastic solids. Jl. Mech. Phys. Solids, 6, 236-249, 1958.

2. R. Hill. Some basic principles in the mechanics of solids without a natural time. Jl. Mech. Phys. Solids, 7, 209-225, 1959.

3. R. Hill. Eigenmodal deformations in elastic/plastic continua. Jl. Mech. Phys. Solids, 15, 371-386, 1967.

4. J. R. Rice. A note on the "small strain" formulation for elastic-plastic problems. Technical Report N00014-67-A-0191-0003/8, Division of Engineering, Brown University, 1970.

5. R. M. McMeeking and J. R. Rice. Finite-element formulations for problems of large elastic-plastic deformation, Int. Jl. Solids Structures, 11, 601-616, 1975.

6. Y. C. Fung. Foundations of solid mechanics, Prentice-Hall, 1965.

7. C. Truesdell and W. Noll. The non-linear field theories of mechanics. Handbuch der Physik, Band III/3, Springer-verlag, 1965.

8. W. Prager, Introduction to mechanics of continua, Ginn and Co., 1961.

9. E. H. Lee and T. Wierzbicki. Analysis of the propagation of plane elastic-plastic wave of finite strain. Jl. Appl. Mech., 34, 931-936, 1967.

10. E. H. Lee. Elastic-plastic deformation at finite strains. Jl. Appl. Mech., 36, 1-6, 1969.

Appendix

For simplicity, relations are developed for Cartesian axes in the reference state which is instantaneously coincident with the current state. The general theory for convected coordinates carries through in a similar manner but can be technically much more involved.

Equations (16) and (24) yield the relation,

$$\underset{\sim}{\tau}\,\underset{\sim}{F}^{T} = \underset{\sim}{s} \tag{A1}$$

Material differentiation of this relation, which is equivalent to convected differentiation, gives

$$\dot{\underset{\sim}{\tau}}\,\underset{\sim}{F}^{T} + \tau\,\dot{F}^{T} = \dot{s}$$

and for the special reference configuration

$$\underset{\sim}{F} = \underset{\sim}{1}\ , \quad \dot{F} = \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_j}\ , \quad \text{hence}$$

$$\dot{s}_{ij} = \dot{\tau}_{ij} + \tau_{ik}\,\frac{\partial v_j}{\partial x_k} \tag{A2}$$

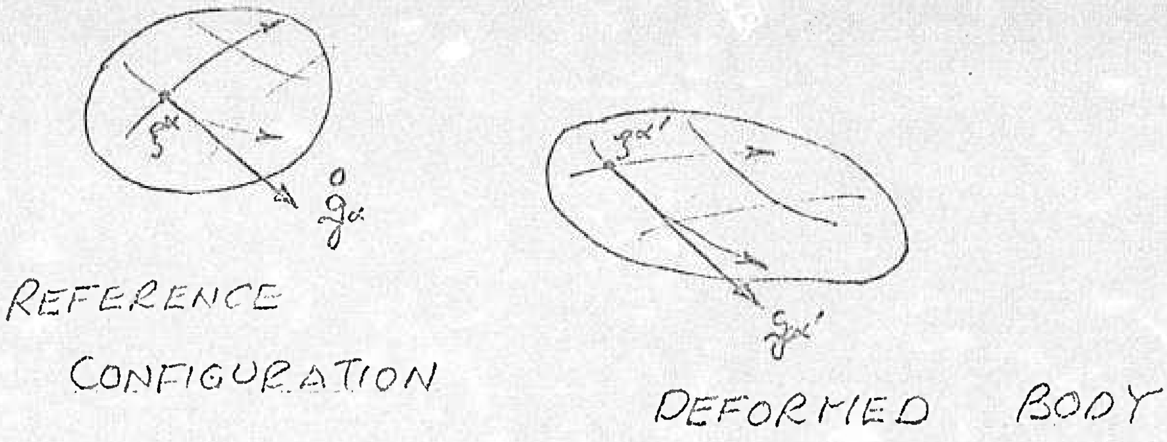

FIG. 1. CONVECTED COORDINATES, $\xi^{\alpha} = \xi^{\alpha'}$.

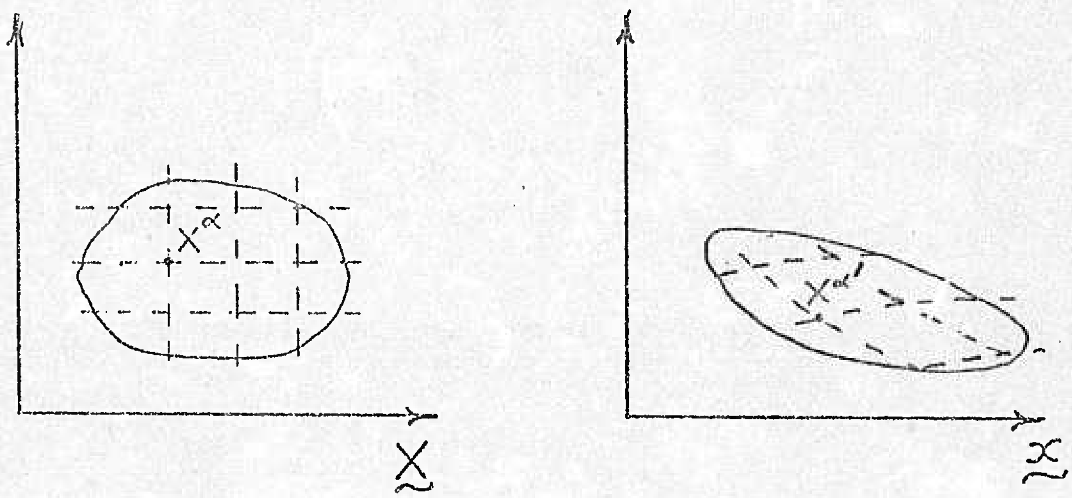

FIG. 2. CARTESIAN REFERENCE COORDINATES, X^{α}